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Particle on a conical surface

M. Merad, T. Boudjedaa, L. Chetouani

Département de Physique, Faculté des Sciences, Université Mentouri, 25000 Constantine, Algeria

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Abstract. The motion of a particle on the conical surface interacting with a scalar and a vector potential is studied in the path integral framework following the technique of constraints. The propagators are evaluated taking into account the problem of the angle periodicity. The result is given in a compact form.

1 Introduction

The path integral technique remains, without doubt, the most efficacious tool used in treating quantum problems. The virtue of this method lies in the fact that it can elegantly fit all situations and can be handled with ease in the case of approximations. In order to do this, the evolution problems of Schrödinger type have been converted to expressions in terms of a path integral. The same task has been extended to the case of Klein–Gordon type by using the Schwinger proper time and to field theory by introducing the Lagrangian density of the field. Unfortunately, in these cases some difficulties subsist because of parametrization and gauge invariance. As is well known, these difficulties occur in the context of quantization with constraints and for these physical systems the evolution equation is not sufficient for a complete description. In general, we have to supply it with some constraints on the state space to ensure the good interpretability of the theory. This manner of quantization is tantamount to saying that the effective dynamics of the system happens in the subspace of the phase space defined by the constraint equations. In consequence, the phase space is reduced and this reduction is reflected in the formalism by the redefinition of some new brackets of evolution named Dirac brackets [1]. In the path integral framework, all this change is introduced by means of a delta functional which allows us to project the dynamics on the reduced phase space. This technique is known as the Faddeev–Senjanovic formulation [2].

The aim of this paper is to study, via the path integral approach, the dynamics of a non-relativistic charged particle without spin and constrained to move on a cone surface [3] and moreover subject to some simple interactions like

(1) the magnetic field,
(2) the harmonic oscillator, and
(3) the Coulomb potential.

In fact, the study of this physical problem turns out to be very interesting because it sets up a link between the non-local effects of quantum mechanics and the geometric singularity [4]. For example, in our case the singularity of the cone at the origin is reflected on the dynamics of the particle by modifying its angular momentum. This is equivalent to the non-local physical effect of Aharanov–Bohm.

In Sect. 2, we expose the general feature of the path integral on the cone surface by considering the case of an unspecified variety [5,6]. In order to do that, we make use of the technique of the constraints worked out by Faddeev–Senjanovic. This consists principally of introducing these constraints by means of a Dirac functional calculus which amounts to a projection of the dynamics of the system on the reduced phase space [1]. Then we integrate over the momentum variables by means of the delta functional, which allows the propagator to be expressed in the configuration space. In the case of the cone the constraint of the surface is eliminated by resorting to polar coordinates which play the role of intrinsic coordinates. We present some applications of this technique related to the elementary physical cases in which the propagator is expressed in the covering space. The final result of the propagator or the Green's function depends on the arbitrary phase. Section 3 is devoted to concluding remarks.

2 General formalism

Let us consider a charged particle subject to the action of scalar and vectorial potentials and moving on the surface of a cone. The Hamiltonian governing the dynamics of this physical system is written

$$\mathcal{H}_T = \frac{\boldsymbol{\pi}^2}{2m} - \lambda f(\boldsymbol{x}) + v p_\lambda + V(\boldsymbol{x}), \tag{1}$$

where $\boldsymbol{\pi} = (\boldsymbol{p} - e\boldsymbol{A}(\boldsymbol{x}))$ and $\boldsymbol{x}, \boldsymbol{p}$ and $\boldsymbol{A}$ are vectors of dimension D. λ is the Lagrange multiplier.

At this stage, let us apply the Dirac procedure. Thus, the involved constraints are

$$\phi_1 = p_\lambda = 0, \tag{2}$$

$$\phi_2(\boldsymbol{x}) = f(\boldsymbol{x}) \simeq 0, \tag{3}$$

$$\phi_3(\boldsymbol{p},\boldsymbol{x}) = \{\phi_2, \mathcal{H}_T\} = \frac{1}{m}\pi_\mu \partial^\mu f(\boldsymbol{x}) \simeq 0, \tag{4}$$

$$\begin{aligned}\phi_4(\boldsymbol{p},\boldsymbol{x},\lambda) &= \{\phi_3, \mathcal{H}_T\} = \frac{1}{m^2}\pi_\mu\pi_\nu\partial^\mu\partial^\nu f(\boldsymbol{x}) + \frac{\lambda}{m}\partial^\nu f(\boldsymbol{x})\partial_\nu f(\boldsymbol{x}) \\ &+ \frac{e}{m^2}\pi^\nu\partial^\mu f(\boldsymbol{x})F_{\mu\nu}(\boldsymbol{x}) - \frac{1}{m}\partial^\nu f(\boldsymbol{x})\partial_\nu V(\boldsymbol{x}),\end{aligned} \tag{5}$$

where $F_{\mu\nu}(\boldsymbol{x}) = \partial_\mu A_\nu(\boldsymbol{x}) - \partial_\nu A_\mu(\boldsymbol{x})$.

The determinant of $\{\phi^a, \phi^b\}$ does not vanish and this consequently indicates that the type of constraints is of the second class. According to the technique of Faddeev–Senjanovic, the propagator related to the problem is written

$$\begin{aligned}K(f,i;T) = &\int \prod_{j=1}^{N} \mathrm{d}\boldsymbol{x}_j \prod_{j=1}^{N+1} \frac{\mathrm{d}\boldsymbol{p}_j}{(2\pi)^{(D-1)}} \mathrm{d}\lambda_j \mathrm{d}p_{\lambda j} \\ &\times \delta(p_{\lambda j})\,\delta(f(\boldsymbol{x}_j))\,\delta(\phi_3(\boldsymbol{p}_j,\overline{\boldsymbol{x}}_j)) \\ &\times \prod_{j=1}^{N+1} \delta(\phi_4(\boldsymbol{p}_j,\overline{\boldsymbol{x}}_j,\lambda_j)\sqrt{\det\{\phi^a,\phi^b\}} \\ &\times \exp\left[\mathrm{i}\left(\boldsymbol{p}_j \Delta\boldsymbol{x}_j + p_{\lambda j}\Delta\lambda_j - \varepsilon\mathcal{H}_T\right)\right],\end{aligned} \tag{6}$$

where the constraints ϕ_3 and ϕ_4 are evaluated at the midpoint with $\overline{\boldsymbol{x}} = (\boldsymbol{x}(t_j) + \boldsymbol{x}(t_{j+1}))/2$ [7]. This choice proves to be, for the moment, necessary since it is not yet clear how to handle the unspecified point in the case of the constraints. The integration on the λ_j and p_{λ_j} variables is immediate and the infinite constants which result from it are absorbed in a redefinition of the measure. Consequently, the result is reduced to

$$\begin{aligned}K(f,i;T) = &\int \prod_{j=1}^{N} \mathrm{d}\boldsymbol{x}_j \prod_{j=1}^{N+1} \frac{\mathrm{d}\boldsymbol{p}_j}{(2\pi)^{(D-1)}} \\ &\times \delta(f(\boldsymbol{x}_j))\,\delta(\phi_3(\boldsymbol{p}_j,\overline{\boldsymbol{x}}_j))\left|\{\phi_2(\boldsymbol{x}_j),\phi_3(\boldsymbol{p}_j,\overline{\boldsymbol{x}}_j)\}\right| \\ &\times \prod_{j=1}^{N+1} \exp\left[\mathrm{i}\left(\boldsymbol{p}_j\Delta\boldsymbol{x}_j - \varepsilon\frac{\boldsymbol{\pi}^2}{2m} - \varepsilon V(\boldsymbol{x}_j)\right)\right],\end{aligned} \tag{7}$$

where

$$\phi_3(\boldsymbol{p}_j,\overline{\boldsymbol{x}}_j) = \frac{1}{m}\left(p_{\mu_j} - eA_\mu(\overline{\boldsymbol{x}}_j)\right)\partial^\mu f(\overline{\boldsymbol{x}}_j), \tag{8}$$

and

$$\{\phi_2(\boldsymbol{x}_j),\phi_3(\boldsymbol{p}_j,\overline{\boldsymbol{x}}_j)\} = \partial_\mu f(\boldsymbol{x}_j)\partial^\mu f(\overline{\boldsymbol{x}}_j). \tag{9}$$

To be able to integrate over the variables let us introduce the integral representation of the delta function:

$$\begin{aligned}&\delta(\phi_3(\boldsymbol{p}_j,\overline{\boldsymbol{x}}_j)) \\ &= \frac{1}{2\pi}\int \mathrm{d}\chi_j \exp\left[\mathrm{i}\chi_j\frac{1}{m}\left(p_{\mu j} - eA_\mu(\overline{x}_j)\right)\partial^\mu f(\overline{x}_j)\right];\end{aligned} \tag{10}$$

then making the change $\boldsymbol{P} = \boldsymbol{p} - e\boldsymbol{A}(\overline{\boldsymbol{x}}_j)$, one obtains

$$\begin{aligned}K(f,i;T) = &\left(\frac{m}{2\pi\mathrm{i}\varepsilon}\right)^{(D-1)(N+1)/2} \\ &\times \int \prod_{j=1}^{N} \mathrm{d}\boldsymbol{x}_j \prod_{j=1}^{N+1} \delta(f(\boldsymbol{x}_j))\sqrt{\partial_\mu f(\boldsymbol{x}_j)\partial^\mu f(\overline{\boldsymbol{x}}_j)} \\ &\times \exp\left[\mathrm{i}\left(\frac{m}{2\varepsilon}\Delta\boldsymbol{x}_j\boldsymbol{\Omega}\Delta\boldsymbol{x}_j + e\Delta\boldsymbol{x}_j\boldsymbol{A}(\overline{\boldsymbol{x}}_j) - \varepsilon V(\boldsymbol{x}_j)\right)\right],\end{aligned} \tag{11}$$

where $\boldsymbol{\Omega}$ is a matrix defined by the following elements $\boldsymbol{\Omega}_{\mu\nu} = \delta_{\mu\nu} - \eta_\mu\eta_\nu$, with $\boldsymbol{\eta}$ the vector

$$\eta_\mu = \frac{\partial_\mu f(\overline{\boldsymbol{x}})}{\sqrt{\left(\dfrac{\partial f}{\partial\overline{\boldsymbol{x}}}\right)^2}}. \tag{12}$$

We can conclude that all the corrections brought about by the constraint are gathered in the $(\partial_\mu f(\boldsymbol{x}_j)\partial^\mu f(\overline{\boldsymbol{x}}_j))^{1/2}$ term.

For the case of the cone surface, the dimension of the space is fixed to $D = 3$ and the function $f(\boldsymbol{x})$, ($\boldsymbol{x} = x,y,z$), is given by

$$f(\boldsymbol{x}) \equiv z - \sqrt{\alpha^{-2}-1}\sqrt{x^2+y^2} = 0, \tag{13}$$

α being a parameter connected to the half aperture γ of the cone by $\alpha = \sin\gamma$. The Poisson bracket $\{\phi_2(\boldsymbol{x}_j), \phi_3(\boldsymbol{p}_j,\overline{\boldsymbol{x}}_j)\}$ is then easily evaluated and the propagator is written

$$\begin{aligned}K(f,i;T) = &\lim_{N\longrightarrow\infty}\int \prod_{j=1}^{N} \mathrm{d}\boldsymbol{x}_j \prod_{j=1}^{N+1} \delta(f(\boldsymbol{x}_j))\left(\frac{m\alpha}{2\pi\mathrm{i}\varepsilon}\right) \\ &\times \left(1 + \left(\alpha^{-2}-1\right)\frac{x_j\overline{x}_j + y_j\overline{y}_j}{\sqrt{x_j^2+y_j^2}\sqrt{\overline{x}_j^2+\overline{y}_j^2}}\right) \\ &\times \prod_{j=1}^{N+1} \exp\left[\mathrm{i}\left(\frac{m}{2\varepsilon}\Delta\boldsymbol{x}_j\boldsymbol{\Omega}\Delta\boldsymbol{x}_j + e\Delta\boldsymbol{x}_j\boldsymbol{A}(\overline{\boldsymbol{x}}_j) - \varepsilon V(\boldsymbol{x}_j)\right)\right],\end{aligned} \tag{14}$$

where $\boldsymbol{\Omega}$ takes the following form:

$$\boldsymbol{\Omega}_{\mu\nu} = \begin{pmatrix} \dfrac{\overline{y}_j^2 + \alpha^2\overline{x}_j^2}{\overline{x}_j^2+\overline{y}_j^2} & \dfrac{\left(\alpha^2-1\right)\overline{x}_j\overline{y}_j}{\overline{x}_j^2+\overline{y}_j^2} & \dfrac{\alpha\sqrt{1-\alpha^2}\,\overline{x}_j}{\sqrt{\overline{x}_j^2+\overline{y}_j^2}} \\ \dfrac{\left(\alpha^2-1\right)\overline{x}_j\overline{y}_j}{\overline{x}_j^2+\overline{y}_j^2} & \dfrac{\overline{x}_j^2 + \alpha^2\overline{y}_j^2}{\overline{x}_j^2+\overline{y}_j^2} & \dfrac{\alpha\sqrt{1-\alpha^2}\,\overline{y}_j}{\sqrt{\overline{x}_j^2+\overline{y}_j^2}} \\ \dfrac{\alpha\sqrt{1-\alpha^2}\,\overline{x}_j}{\sqrt{\overline{x}_j^2+\overline{y}_j^2}} & \dfrac{\alpha\sqrt{1-\alpha^2}\,\overline{y}_j}{\sqrt{\overline{x}_j^2+\overline{y}_j^2}} & 1-\alpha^2 \end{pmatrix}. \tag{15}$$

In the following we are concerned with the computations of the expression (14) in the particular cases cited above. To illustrate this quantization method, let us first present the explicit calculations relative to the case of a free particle ($\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{0}$, $V(\boldsymbol{x}) = \boldsymbol{0}$).

The Lagrangian of this system is then written as

$$\mathcal{L} = \frac{m}{2}\,\dot{\boldsymbol{x}}^2 + \lambda(z - \sqrt{\alpha^{-2}-1}\sqrt{x^2+y^2}), \tag{16}$$

where λ is the Lagrange multiplier which imposes the particle to be on the cone surface.

The equations (2)–(5) now take the following simplified forms:

$$\phi_{1j} = \frac{\partial \mathcal{L}}{\partial \dot{\lambda}} = v = p_{\lambda j} = 0, \tag{17}$$

$$\phi_{2j} = z_j - \sqrt{\alpha^{-2}-1}\sqrt{x_j^2+y_j^2} \simeq 0, \tag{18}$$

$$\phi_{3j} = \{\phi_{2j}, \mathcal{H}_T\} = \frac{1}{m}\left(p_{zj} - \frac{\sqrt{\alpha^{-2}-1}}{\sqrt{\overline{x}_j^2+\overline{y}_j^2}}\left(\overline{x}_j p_{xj} + \overline{y}_j p_{yj}\right)\right) \simeq 0, \tag{19}$$

$$\phi_{4j} = \{\phi_{3j}, \mathcal{H}_T\} = \frac{\alpha^{-2}\lambda}{m} - \frac{1}{m^2}\frac{\sqrt{\alpha^{-2}-1}}{\sqrt{\left(\overline{x}_j^2+\overline{y}_j^2\right)^3}}\left(\overline{y}_j^2 {p_{xj}}^2 + \overline{x}_j^2 {p_{yj}}^2\right) \simeq 0. \tag{20}$$

Substituting these constraints in the propagator (7), we get the result

$$\begin{aligned}\mathcal{K}(f,i;T) = \lim_{N\longrightarrow\infty}\int \prod_{j=1}^{N} \mathrm{d}\boldsymbol{x}_j \prod_{j=1}^{N+1} \frac{\mathrm{d}\boldsymbol{p}_j}{(2\pi)^2} &\times \delta\left(z_j - \sqrt{\alpha^{-2}-1}\sqrt{x_j^2+y_j^2}\right) \\ &\times \left(1 + \frac{\left(\alpha^{-2}-1\right)\left(x_j\overline{x}_j + y_j\overline{y}_j\right)}{\sqrt{x_j^2+y_j^2}\sqrt{\overline{x}_j^2+\overline{y}_j^2}}\right) \\ &\times \delta\left(p_{z_j} - \frac{\sqrt{\alpha^{-2}-1}\left(\overline{x}_j p_{xj} + \overline{y}_j p_{yj}\right)}{\sqrt{{\overline{x}_j}^2 + {\overline{y}_j}^2}}\right) \\ &\times \prod_{j=1}^{N+1} \exp\left\{\mathrm{i}\left[\boldsymbol{p}_j \Delta\boldsymbol{x}_j - \frac{\varepsilon \boldsymbol{p}_j^2}{2m}\right]\right\}.\end{aligned} \tag{21}$$

Following step by step the previous presentation, we will obtain the following result:

$$\begin{aligned}\mathcal{K}(f,i;T) = &\int \prod_{j=1}^{N} \mathrm{d}\boldsymbol{x}_j \prod_{j=1}^{N+1}\left(\frac{m\alpha}{2\pi\mathrm{i}\varepsilon}\right) \\ &\times \delta(z_j - \sqrt{\alpha^{-2}-1}\sqrt{x_j^2+y_j^2}) \\ &\times \left(1 + \frac{\left(\alpha^{-2}-1\right)\left(x_j\overline{x}_j + y_j\overline{y}_j\right)}{\sqrt{x_j^2+y_j^2}\sqrt{\overline{x}_j^2+{\overline{y}_j}^2}}\right) \\ &\times \prod_{j=1}^{N+1} \exp\left[\frac{\mathrm{i}m}{2\varepsilon}\left(\Delta x_j^2 + \Delta y_j^2 + \Delta z_j^2 \right.\right. \\ &\left.\left. -\ \alpha^2\left(\sqrt{\alpha^{-2}-1}\frac{\overline{x}_j\Delta x_j + \overline{y}_j\Delta y_j}{\sqrt{{\overline{x}_j}^2+\overline{y}_j^2}} - \Delta z_j\right)^2\right)\right].\end{aligned} \tag{22}$$

Now, it is suitable to introduce the polar coordinates (r,θ) which are in fact the intrinsic coordinates of the cone surface. Furthermore, this change allows us to integrate along the z paths and at the same time decouples the variables x and y. In effect, putting $x = r\cos\theta$ and $y = r\sin\theta$, the measure becomes $\mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\theta$ and the determinant of the constraints gives

$$1 + \left(\alpha^{-2}-1\right)\frac{x_j\overline{x}_j + y_j\overline{y_j}}{\sqrt{x_j^2+y_j^2}\sqrt{{\overline{x_j}}^2+\overline{y}_j^2}} \simeq \alpha^{-2}\left[1 - \frac{1-\alpha^2}{8}\Delta\theta_j^2\right]. \tag{23}$$

Let us symmetrize the measure and then develop it together with the infinitesimal action to the order ε in the vicinity of the mid-point:

$$\prod_{j=1}^{N} r_j = \frac{1}{\sqrt{r_i r_f}}\prod_{j=1}^{N+1}\sqrt{r_j r_{j-1}} \simeq \frac{1}{\sqrt{r_i r_f}}\prod_{j=1}^{N+1}\overline{r_j}\left(1 - \frac{\Delta r_j^2}{8{\overline{r_j}}^2}\right), \tag{24}$$

and

$$\exp\left\{-\mathrm{i}\left[\frac{m}{8\varepsilon}\left(\Delta r_j^2\Delta\theta_j^2 + \frac{1}{3}{\overline{r_j}}^2\Delta\theta_j^4\right)\right]\right\} \simeq 1 - \mathrm{i}\frac{m}{8\varepsilon}\left(\Delta r_j^2\Delta\theta_j^2 + \frac{1}{3}{\overline{r_j}}^2\Delta\theta_j^4\right). \tag{25}$$

By retaining only the terms contributing in the action and by substituting the suitable equations in the expression of the propagator (21) and using the property $f(z)\delta(z-a) = f(a)\delta(z-a)$, the propagator then takes the following form:

$$\begin{aligned}K(f,i;T) &= \frac{1}{\sqrt{r_i r_f}}\int \prod_{j=1}^{N}\mathrm{d}r_j\mathrm{d}\theta_j\mathrm{d}z_j \prod_{j=1}^{N+1}\delta(z_j - \sqrt{\alpha^{-2}-1}r_j) \\ &\times \left(\frac{m\overline{r_j}}{2\pi\mathrm{i}\varepsilon\alpha}\right)(1+C_T)\exp\left\{\mathrm{i}\frac{m}{2\varepsilon}\left(\alpha^{-2}\Delta r_j^2 + {\overline{r_j}}^2\Delta\theta_j^2\right)\right\},\end{aligned} \tag{26}$$

where C_T represents the total correction resulting from to the mid-point development of the measure, and from the determinant constraints and the infinitesimal action. This total correction is given as

$$\begin{aligned}(1+C_T) &= \left(1 - \frac{\Delta r_j^2}{8{\overline{r_j}}^2}\right)\left(1 - \frac{1-\alpha^2}{8}\Delta\theta_j^2\right) \\ &\times \left(1 - \mathrm{i}\frac{m}{8\varepsilon}\left(\Delta r_j^2\Delta\theta_j^2 + \frac{1}{3}{\overline{r_j}}^2\Delta\theta_j^4\right)\right) \\ &= 1 - \frac{\Delta r_j^2}{8{\overline{r_j}}^2} - \frac{1-\alpha^2}{8}\Delta\theta_j^2 - \mathrm{i}\frac{m}{8\varepsilon}\Delta r_j^2\Delta\theta_j^2 - \mathrm{i}\frac{m}{24\varepsilon}{\overline{r_j}}^2\Delta\theta_j^4,\end{aligned} \tag{27}$$

where only the terms contributing to order 1 in ε were retained.

Consequently, according to the usual procedure, the additional corrections are replaced by a purely quantum effective potential [8]. Carrying out the integration over the variables z, we get the following result:

$$K(f,i;T)=\frac{1}{\sqrt{r_f r_i}}\int rDrD\theta \times \exp\left[\mathrm{i}\int_0^T \mathrm{d}t\left(\frac{m}{2}\left(\alpha^{-2}\,\dot{r}^2+r^2\,\dot{\theta}^2\right)+\frac{\alpha^2}{8mr^2}\right)\right]. \tag{28}$$

In order to absorb the parameter α appearing in the kinetic term, let us introduce the following changes [9]:

$$\alpha^{-1}r=\rho \quad \text{and} \quad \alpha\theta=\varphi, \tag{29}$$

where $0\le\varphi\le 2\pi\alpha$.

With these new variables, the relative propagator is expressed in this case as

$$K(f,i;T)=\frac{1}{\alpha}\frac{1}{\sqrt{\rho_f\rho_i}}\int \rho D\rho D\varphi \times \exp\left[\mathrm{i}\int_0^T \mathrm{d}t\left(\frac{m}{2}\left(\dot{\rho}^2+\rho^2\,\dot{\varphi}^2\right)+\frac{1}{8m\rho^2}\right)\right]. \tag{30}$$

Knowing that the angle is of $2\pi\alpha$ period, it is suitable to take this into consideration by using the covering space expression of the propagator. To do this, we develop the propagator on the series of propagators associated with trajectories winding around the origin n times, namely

$$K(f,i;T)=\sum_{n=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}n\delta}K_n(f,i;T), \tag{31}$$

where $K_n(f,i;T)$ is the propagator associated with the winding number n:

$$K_n(f,i;T)=\frac{1}{\alpha}\frac{1}{\sqrt{\rho_f\rho_i}}\int_{\rho_i,\varphi_i}^{\rho_f,\varphi_f+2\pi n\alpha}\rho D\rho D\varphi \times \exp\left[\mathrm{i}\int_0^T \mathrm{d}t\left(\frac{m}{2}\left(\dot{\rho}^2+\rho^2\,\dot{\varphi}^2\right)+\frac{1}{8m\rho^2}\right)\right], \tag{32}$$

where $0<\rho<\infty$ and $-\infty<\varphi<\infty$.

At this stage, it is easy to integrate over the φ variables to get the result

$$K_n(f,i;T)=\frac{1}{\alpha}\frac{1}{\sqrt{\rho_f\rho_i}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}p}{2\pi}\mathrm{e}^{\mathrm{i}p(\varphi_f-\varphi_i+2\pi n\alpha)}\int D\rho \times \exp\left[\mathrm{i}\int_0^T \mathrm{d}t\left(\frac{m}{2}\,\dot{\rho}^2-\frac{4p^2-1}{8m\rho^2}\right)\right]. \tag{33}$$

Then inserting this in (33) and using the Poisson identity, we get for the propagator

$$K_n(f,i;T)=\frac{1}{\alpha}\frac{1}{\sqrt{\rho_f\rho_i}}\sum_{l=-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{d}\lambda \times \int_{-\infty}^{+\infty}\frac{\mathrm{d}p}{2\pi}\mathrm{e}^{\mathrm{i}p(\varphi_f-\varphi_i)+\mathrm{i}\lambda(2\pi\alpha p-2\pi l+\delta)} \times \int D\rho\exp\left[\mathrm{i}\int_0^T \mathrm{d}t\left(\frac{m}{2}\,\dot{\rho}^2-\frac{4p^2-1}{8m\rho^2}\right)\right]. \tag{34}$$

The integration over the variable λ gives $\delta(p-(l/\alpha)+\delta/(2\pi\alpha))$ which permits us to simplify the p variable and reduce the propagator to that of the centrifugal harmonic oscillator with orbital momentum equal to $(\mid l-\delta/2\pi\mid)/\alpha$

$$K(f,i;T)=\left(\frac{m}{2\pi\mathrm{i}\alpha^2T}\right)\times\exp\left[\frac{\mathrm{i}m}{2\alpha^2T}\left(r_f^2+r_i^2\right)-\mathrm{i}\frac{\delta}{2\pi}\left(\theta_f-\theta_i\right)\right]\times\sum_{l=-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}l(\theta_f-\theta_i)}I_{|l-\delta/2\pi|/\alpha}\left(\frac{mr_fr_i}{\alpha^2\mathrm{i}T}\right). \tag{35}$$

In order to deduce the wave functions from this expression, let us use the following formula [10]:

$$\int_0^\infty \mathrm{d}x\exp(-\omega x)J_\nu\left(2\beta\sqrt{x}\right)J_\nu\left(2\gamma\sqrt{x}\right)=\frac{1}{\omega}\exp\left[-\frac{1}{\omega}\left(\beta^2+\gamma^2\right)\right]I_\nu\left(\frac{2\beta\gamma}{\omega}\right), \tag{36}$$

to separate the variables r_f, r_i and T.

Then the propagator takes the following form:

$$K(f,i;T)=\frac{m}{\alpha^2}\sum_{l=-\infty}^{+\infty}\frac{1}{2\pi}\mathrm{e}^{\mathrm{i}(l-\delta/2\pi)(\theta_f-\theta_i)}\int_0^\infty \mathrm{d}E\mathrm{e}^{-\mathrm{i}ET}\times J_{|l-\delta/2\pi|/\alpha}\left(\sqrt{\frac{2mE}{\alpha^2}}r_f\right)J_{|l-\delta/2\pi|/\alpha}\left(\sqrt{\frac{2mE}{\alpha^2}}r_i\right), \tag{37}$$

and this allows us to extract the normalized energy depending wave functions $(0\le E\le\infty)$:

$$\psi_{l,E}(r,\theta)=\sqrt{\frac{m}{2\pi\alpha^2}}\mathrm{e}^{\mathrm{i}(l-\delta/2\pi)\theta}J_{|l-\delta/2\pi|/\alpha}\left(\sqrt{\frac{2mE}{\alpha^2}}r\right). \tag{38}$$

In the end, let us notice that the result is not periodic and generally depends on the phase δ which is fixed by hand. In order to write the propagator in a compact form, let us separate it in two parts: one discrete and finite and another continuous. To this aim let us do the summation of the series present in the expression (35) by introducing the following the integral representation of the Bessel function [10]:

$$I_\nu(z)=\frac{1}{\pi}\int_0^\pi\exp(z\cos\phi)\cos(\nu\phi)\mathrm{d}\phi-\frac{\sin\nu\pi}{\pi}\int_0^\infty\exp(-z\cosh t-\nu t)\mathrm{d}t, \quad \mid\arg z\mid\le\pi/2,\quad \mathrm{Re}\,\nu\ge 0. \tag{39}$$

In effect, inserting this in expression (35) we get the following result:

$$K(f,i;T)=\left(\frac{m}{2\pi\mathrm{i}\alpha^2T}\right)\exp\left[\frac{\mathrm{i}m}{2\alpha^2T}\left(r_f^2+r_i^2\right)-\mathrm{i}\frac{\delta}{2\pi}\Delta\theta\right]\times\left[\frac{1}{\pi}\int_0^{\pi}\mathrm{d}\phi\exp(z\cos\phi)\mathcal{S}_1+\frac{1}{\pi}\left(\lim_{\varepsilon\to0^+}\int_0^{\varepsilon}\mathrm{d}t\exp(-z\cosh\varepsilon)\mathcal{S}_2+\int_{\varepsilon}^{\infty}\mathrm{d}t\exp(-z\cosh t)\mathcal{S}_3\right)\right],\tag{40}$$

with $\mathcal{S}_1,\mathcal{S}_2$ and $\mathcal{S}_3$ as the series defined by

$$\mathcal{S}_1=\sum_{l=-\infty}^{\left[\frac{\delta}{2\pi}\right]}\mathrm{e}^{\mathrm{i}l\Delta\theta}\cos\left(\frac{l-\delta/2\pi}{\alpha}\phi\right)+\sum_{l=\left[\frac{\delta}{2\pi}\right]+1}^{+\infty}\mathrm{e}^{\mathrm{i}l\Delta\theta}\cos\left(\frac{l-\delta/2\pi}{\alpha}\phi\right),\tag{41}$$

$$\mathcal{S}_2=\sum_{l=-\infty}^{\left[\frac{\delta}{2\pi}\right]}\mathrm{e}^{\mathrm{i}l\Delta\theta}\sin\left(\frac{l-\delta/2\pi}{\alpha}\pi\right)\exp\left(\frac{l-\delta/2\pi}{\alpha}\varepsilon\right)-\sum_{l=\left[\frac{\delta}{2\pi}\right]+1}^{+\infty}\mathrm{e}^{\mathrm{i}l\Delta\theta}\sin\left(\frac{l-\delta/2\pi}{\alpha}\pi\right)\exp\left(-\frac{l-\delta/2\pi}{\alpha}\varepsilon\right)\tag{42}$$

and

$$\mathcal{S}_3=\sum_{l=-\infty}^{\left[\frac{\delta}{2\pi}\right]}\mathrm{e}^{\mathrm{i}l\Delta\theta}\sin\left(\frac{l-\delta/2\pi}{\alpha}\pi\right)\exp\left(\frac{l-\delta/2\pi}{\alpha}t\right)-\sum_{l=\left[\frac{\delta}{2\pi}\right]+1}^{+\infty}\mathrm{e}^{\mathrm{i}l\Delta\theta}\sin\left(\frac{l-\delta/2\pi}{\alpha}\pi\right)\exp\left(-\frac{l-\delta/2\pi}{\alpha}t\right),\tag{43}$$

where $z=mr_fr_i/(\alpha^2\mathrm{i}T)$, $\Delta\theta=\theta_f-\theta_i$ and $[\delta/2\pi]$ is the integer part of $\delta/2\pi$. We have also decomposed the integral $\int_0^{\infty}=\lim_{\varepsilon\to0^+}\int_0^{\varepsilon}+\int_{\varepsilon}^{\infty}$ because in the domain $[\varepsilon,\infty]$ the series can be summed, whereas in the domain $[0,\varepsilon]$ the sum of series diverges and gives a delta function behavior as we are going to see.

Knowing that all the series are of geometric type, it is easy to compute their sums:

$$\mathcal{S}_1=\pi\alpha\exp\left(-\mathrm{i}\frac{\delta\phi}{2\pi\alpha}\right)\sum_{n=-\infty}^{\infty}\delta(\phi+\alpha\Delta\theta-2\pi n\alpha)+\pi\alpha\exp\left(\mathrm{i}\frac{\delta\phi}{2\pi\alpha}\right)\sum_{n=-\infty}^{\infty}\delta(-\phi+\alpha\Delta\theta-2\pi n\alpha),\tag{44}$$

$$\mathcal{S}_2=\frac{1}{2\mathrm{i}}\mathrm{e}^{-\mathrm{i}\delta/(2\alpha)+\mathrm{i}[\delta/2\pi](\Delta\theta+(\pi/\alpha))}\times\left[\frac{\alpha\{\delta/2\pi\}\left(\Delta\theta+\frac{\pi}{\alpha}\right)^2\varepsilon+2\mathrm{i}\alpha^2\sin\left(\Delta\theta+\frac{\pi}{\alpha}\right)}{4\alpha^2\sin^2\left(\frac{\Delta\theta+(\pi/\alpha)}{2}\right)+\varepsilon^2}-\left(1+\{\delta/2\pi\}\frac{\varepsilon}{\alpha}\right)\right]-\frac{1}{2\mathrm{i}}\mathrm{e}^{\mathrm{i}\delta/(2\alpha)+\mathrm{i}[\delta/2\pi](\Delta\theta-(\pi/\alpha))}\times\left[\frac{\alpha\{\delta/2\pi\}\left(\Delta\theta-\frac{\pi}{\alpha}\right)^2\varepsilon+2\mathrm{i}\alpha^2\sin\left(\Delta\theta-\frac{\pi}{\alpha}\right)}{4\alpha^2\sin^2\left(\frac{\Delta\theta-(\pi/\alpha)}{2}\right)+\varepsilon^2}-\left(1+\{\delta/2\pi\}\frac{\varepsilon}{\alpha}\right)\right]\tag{45}$$

and

$$\mathcal{S}_3=\frac{\mathrm{e}^{-\mathrm{i}\delta/2\alpha+\mathrm{i}[\delta/2\pi](\Delta\theta+(\pi/\alpha))}}{2\mathrm{i}}\times\left[\frac{\mathrm{e}^{t/\alpha}\sinh\left(\{\delta/2\pi\}\frac{t}{\alpha}\right)+\sinh\left(\mathrm{i}\left(\Delta\theta+\frac{\pi}{\alpha}\right)-\{\delta/2\pi\}\frac{t}{\alpha}\right)}{\cosh\left(\frac{t}{\alpha}\right)-\cos\left(\Delta\theta+\frac{\pi}{\alpha}\right)}-\mathrm{e}^{\{\delta/2\pi\}(t/\alpha)}\right]-\frac{\mathrm{e}^{\mathrm{i}\delta/2\alpha+\mathrm{i}[\delta/2\pi](\Delta\theta-(\pi/\alpha))}}{2\mathrm{i}}\times\left[\frac{\mathrm{e}^{t/\alpha}\sinh\left(\{\delta/2\pi\}\frac{t}{\alpha}\right)+\sinh\left(\mathrm{i}\left(\Delta\theta-\frac{\pi}{\alpha}\right)-\{\delta/2\pi\}\frac{t}{\alpha}\right)}{\cosh\left(\frac{t}{\alpha}\right)-\cos\left(\Delta\theta-\frac{\pi}{\alpha}\right)}-\mathrm{e}^{\{\delta/2\pi\}(t/\alpha)}\right],$$

where we have used the well-known identity

$$\sum_{l=-\infty}^{\infty}\exp(\mathrm{i}lx)=2\pi\sum_{n=-\infty}^{\infty}\delta(x-2\pi n),\tag{46}$$

and we have only kept the essential part of the sum $\mathcal{S}_2$. The number $\{\delta/2\pi\}$ denotes the decimal part of $\delta/2\pi$, namely, $\{\delta/2\pi\}=\delta/2\pi-[\delta/2\pi]$.

Consequently, inserting all these results in the expression of the propagator (40), one gets

$$K(f,i;T)=\left(\frac{m}{2\pi\mathrm{i}\alpha^2T}\right)\exp\left[\frac{\mathrm{i}m}{2\alpha^2T}\left(r_f^2+r_i^2\right)-\mathrm{i}\frac{\delta}{2\pi}\Delta\theta\right]\times\left\{\alpha\int_0^{\pi}\mathrm{d}\phi\exp(z\cos\phi)\right.$$

$$\times\left[\exp\left(-\mathrm{i}\frac{\delta\phi}{2\pi}\right)\sum_{n=-\infty}^{\infty}\delta(\phi+\alpha\Delta\theta-2\pi n\alpha)\right.$$
$$\left.+\exp\left(\mathrm{i}\frac{\delta\phi}{2\pi}\right)\sum_{n=-\infty}^{\infty}\delta(-\phi+\alpha\Delta\theta-2\pi n\alpha)\right]$$
$$-\frac{\mathrm{e}^{\mathrm{i}[\delta/2\pi]\Delta\theta-z}}{\pi}$$
$$\times\lim_{\varepsilon\to0^+}\left[\mathrm{e}^{-\mathrm{i}\delta/(2\alpha)+\mathrm{i}[\delta/2\pi](\pi/\alpha)}\frac{\alpha^2\sin\left(\Delta\theta+\frac{\pi}{\alpha}\right)\varepsilon}{4\alpha^2\sin^2\left(\frac{\Delta\theta+(\pi/\alpha)}{2}\right)+\varepsilon^2}\right.$$
$$\left.-\ \mathrm{e}^{\mathrm{i}(\delta/2\alpha)-\mathrm{i}[\delta/2\pi](\pi/\alpha)}\frac{\alpha^2\sin\left(\Delta\theta-\frac{\pi}{\alpha}\right)\varepsilon}{4\alpha^2\sin^2\left(\frac{\Delta\theta-(\pi/\alpha)}{2}\right)+\varepsilon^2}\right]$$
$$-\frac{\mathrm{e}^{\mathrm{i}[\delta/2\pi]\Delta\theta}}{2\pi\mathrm{i}}\lim_{\varepsilon\to0^+}\int_{\varepsilon}^{\infty}\mathrm{d}t\mathrm{e}^{-z\cosh t}\left[\mathrm{e}^{-\mathrm{i}(\delta/(2\alpha))+\mathrm{i}[\delta/2\pi](\pi/\alpha)}\right.$$
$$\times\left(\frac{\mathrm{e}^{t/\alpha}\sinh\left(\{\delta/2\pi\}\frac{t}{\alpha}\right)+\sinh\left(\mathrm{i}\left(\Delta\theta+\frac{\pi}{\alpha}\right)-\{\delta/2\pi\}\frac{t}{\alpha}\right)}{\cosh\left(\frac{t}{\alpha}\right)-\cos\left(\Delta\theta+\frac{\pi}{\alpha}\right)}\right.$$
$$\left.-\ \mathrm{e}^{\{\delta/2\pi\}(t/\alpha)}\right)-\mathrm{e}^{\mathrm{i}(\delta/(2\alpha))-\mathrm{i}[\delta/2\pi](\pi/\alpha)}$$
$$\times\left(\frac{\mathrm{e}^{t/\alpha}\sinh\left(\{\delta/2\pi\}\frac{t}{\alpha}\right)+\sinh\left(\mathrm{i}\left(\Delta\theta-\frac{\pi}{\alpha}\right)-\{\delta/2\pi\}\frac{t}{\alpha}\right)}{\cosh\left(\frac{t}{\alpha}\right)-\cos\left(\Delta\theta-\frac{\pi}{\alpha}\right)}\right.$$
$$\left.\left.\left.-\ \mathrm{e}^{\{\delta/2\pi\}(t/\alpha)}\right)\right]\right\}. \tag{47}$$

With the help of the well-known formulae

$$\frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2}\to\delta(x)\quad\text{and}\quad\delta(\sin(x))=\sum_{n=-\infty}^{\infty}\delta(x-n\pi), \tag{48}$$

it is easy to show that the contribution of the calculations for $0<t<\varepsilon$ gives a vanishing term because of the delta property $x\delta(x)=0$.

Accordingly, after integrating over ϕ and using the delta property $f(x)\delta(x)=f(0)\delta(x)$, the final form of the propagator will be

$$K(f,i;T)=\left(\frac{m}{2\pi\mathrm{i}\alpha T}\right)\exp\left(\frac{\mathrm{i}m}{2\alpha^2T}\left(r_f^2+r_i^2\right)\right)$$
$$\times\left\{\sum_n\exp\left[-\mathrm{i}\delta n-\frac{\mathrm{i}mr_fr_i}{\alpha^2T}\cos\left(2n\pi\alpha-\alpha\Delta\theta\right)\right]\right.$$
$$-\frac{\mathrm{e}^{-\mathrm{i}\{\delta/2\pi\}\Delta\theta}}{2\alpha\mathrm{i}\pi}$$
$$\times\int_{0^+}^{+\infty}\mathrm{d}t\exp\left(\frac{\mathrm{i}mr_fr_i}{\alpha^2T}\cosh t\right)\left[\mathrm{e}^{-\mathrm{i}(\delta/(2\alpha))+\mathrm{i}[\delta/2\pi](\pi/\alpha)}\right.$$
$$\times\left(\frac{\mathrm{e}^{t/\alpha}\sinh\left(\{\delta/2\pi\}\frac{t}{\alpha}\right)+\sinh\left(\mathrm{i}\left(\Delta\theta+\frac{\pi}{\alpha}\right)-\{\delta/2\pi\}\frac{t}{\alpha}\right)}{\cosh\left(\frac{t}{\alpha}\right)-\cos\left(\Delta\theta+\frac{\pi}{\alpha}\right)}\right.$$
$$\left.-\ \mathrm{e}^{\{\delta/2\pi\}(t/\alpha)}\right)-\mathrm{e}^{\mathrm{i}(\delta/(2\alpha))-\mathrm{i}[\delta/2\pi](\pi/\alpha)}$$
$$\times\left(\frac{\mathrm{e}^{t/\alpha}\sinh\left(\{\delta/2\pi\}\frac{t}{\alpha}\right)+\sinh\left(\mathrm{i}\left(\Delta\theta-\frac{\pi}{\alpha}\right)-\{\delta/2\pi\}\frac{t}{\alpha}\right)}{\cosh\left(\frac{t}{\alpha}\right)-\cos\left(\Delta\theta-\frac{\pi}{\alpha}\right)}\right.$$
$$\left.\left.-\ \mathrm{e}^{\{\delta/2\pi\}(t/\alpha)}\right)\right], \tag{49}$$

where the sum over n is limited to the values $-1/(2\alpha)+(\Delta\theta)/(2\pi)\leq n\leq 1/(2\alpha)+(\Delta\theta)/(2\pi)$ and the symbol $\int_{0^+}^{+\infty}$ is $\lim_{\varepsilon\to0}\int_{\varepsilon}^{\infty}$, which represents the principal value of the integral.

Let us also remark that the result of the propagator which is made up of two parts,

(1) a finite discrete sum of propagators,
(2) a superposition of continuous sum of propagators, depends on the arbitrary phase δ, while the bounds are independent.

In the end, it is interesting to consider the periodic case, $\delta=2\pi k$ (k integer); the expression of the propagator (50) is then simplified to

$$K(f,i;T)=\left(\frac{m}{2\pi\mathrm{i}\alpha T}\right)\exp\left(\frac{\mathrm{i}m}{2\alpha^2T}\left(r_f^2+r_i^2\right)\right)$$
$$\times\left\{\sum_n\exp\left[-\frac{\mathrm{i}mr_fr_i}{\alpha^2T}\cos\left(2n\pi\alpha-\alpha\Delta\theta\right)\right]\right.$$
$$-\frac{1}{2\alpha\pi}\int_{0^+}^{+\infty}\mathrm{d}t\exp\left[\frac{\mathrm{i}mr_fr_i}{\alpha^2T}\cosh t\right]$$
$$\times\left[\left(\frac{\sin\left(\Delta\theta+\frac{\pi}{\alpha}\right)}{\cosh\left(\frac{t}{\alpha}\right)-\cos\left(\Delta\theta+\frac{\pi}{\alpha}\right)}\right)\right.$$
$$\left.\left.-\left(\frac{\sin\left(\Delta\theta-\frac{\pi}{\alpha}\right)}{\cosh\left(\frac{t}{\alpha}\right)-\cos\left(\Delta\theta-\frac{\pi}{\alpha}\right)}\right)\right]\right\}, \tag{50}$$

where $-1/(2\alpha)+\Delta\theta/(2\pi)\leq n\leq 1/(2\alpha)+\Delta\theta/(2\pi)$.

2.1 Case of constant magnetic field

For this case the direction of the magnetic field is chosen along the oz axis and the gauge is fixed by

$$\boldsymbol{A} \equiv \frac{H}{2}(-y, x, 0) \quad \text{and} \quad V(\boldsymbol{x}) = 0. \tag{51}$$

The additional term of the magnetic field in the formula (14) is converted into polar coordinates:

$$-\frac{eH}{2}\left(y_j \Delta x_j - x_j \Delta y_j\right) = \frac{m\omega}{\alpha} r_j r_{j-1} \sin \Delta\theta_j \simeq \frac{m\omega}{\alpha} \overline{r_j}^2 \Delta\theta_j, \tag{52}$$

with $\omega = |e| H\alpha/(2m)$.

It is easy to check that the expression of the propagator becomes

$$K(f,i;T) = \frac{1}{\sqrt{r_f r_i}} \int r\mathcal{D}r(t)\mathcal{D}\theta(t) \times \exp\left\{ \mathrm{i}\left[\int_0^T \mathrm{d}t \left(\frac{m}{2}\left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + \frac{m\omega}{\alpha} r^2 \dot{\theta} + \frac{\alpha^2}{8mr^2} \right) \right] \right\}. \tag{53}$$

At this stage let us express the propagator in the covering space to take into account the periodicity of the angle. To do this, we develop here again the propagator in the form of a series of propagators associated with trajectories winding around the origin n times:

$$K(f,i;T) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}n\delta} K_n(f,i;T), \tag{54}$$

where

$$K_n(f,i;T) = \frac{1}{\alpha}\frac{1}{\sqrt{\rho_f\rho_i}} \int_{\rho_i,\varphi_i}^{\rho_f,\varphi_f+2\pi n\alpha} \rho\mathcal{D}\rho(t) \times \int \mathcal{D}\varphi(t) \exp\left\{ \mathrm{i}\left[\int_0^T \mathrm{d}t \left(\frac{m}{2}\left(\dot{\rho}^2 + \rho^2 \dot{\varphi}^2 \right) + m\omega\rho^2 \dot{\varphi} + \frac{1}{8m\rho^2} \right) \right] \right\}. \tag{55}$$

Here, we can also proceed as in the free case by integrating over φ. Consequently, the propagator will be given by

$$K(f,i;T) = \frac{1}{\alpha}\frac{1}{\sqrt{\rho_f\rho_i}} \times \sum_{n=-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}n\delta} \int_{-\infty}^{+\infty} \frac{\mathrm{d}p}{2\pi} \mathrm{e}^{\mathrm{i}p(\varphi_f-\varphi_i+\delta+2\pi\alpha n)} \times \int D\rho \exp \mathrm{i}\left\{ \int_0^T \mathrm{d}t \left[\frac{m}{2}\dot{\rho}^2 - \frac{p^2}{2m\rho^2} - \frac{m\omega^2}{2}\rho^2 + p\omega + \frac{1}{8m\rho^2} \right] \right\}. \tag{56}$$

Let us integrate over the variable p by using the Poisson identity, to get the following result:

$$K(f,i;T) = \frac{1}{\alpha}\frac{1}{\sqrt{\rho_f\rho_i}} \times \sum_{l=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}\lambda \int_{-\infty}^{+\infty} \frac{\mathrm{d}p}{2\pi} \mathrm{e}^{\mathrm{i}p(\varphi_f-\varphi_i)+\mathrm{i}\lambda(2\pi\alpha p-2\pi l+\delta)} \times \int D\rho \exp \mathrm{i}\left\{ \int \mathrm{d}t \left[\frac{m}{2}\dot{\rho}^2 - \frac{p^2}{2m\rho^2} + p\omega - \frac{m\omega^2}{2}\rho^2 + \frac{1}{8m\rho^2} \right] \right\}. \tag{57}$$

Consequently, the integration on the λ variable becomes easy and will yield the following result for the propagator:

$$K(f,i;T) = \frac{1}{\alpha^2}\frac{1}{\sqrt{\rho_f\rho_i}} \times \sum_{l=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\mathrm{d}p}{2\pi} \mathrm{e}^{\mathrm{i}p(\varphi_f-\varphi_i)} \delta\left(p - \frac{l}{\alpha} + \frac{\delta}{2\pi\alpha} \right) \times \int D\rho \exp \mathrm{i}\left\{ \int \mathrm{d}t \left[\frac{m}{2}\dot{\rho}^2 - \frac{p^2}{2m\rho^2} + p\omega - \frac{m\omega^2}{2}\rho^2 + \frac{1}{8m\rho^2} \right] \right\}. \tag{58}$$

Now, we are ready to carry out the integrations over the variable p. This gives the following expression for the propagator:

$$K(f,i;T) = \frac{1}{\alpha^2}\frac{1}{\sqrt{\rho_f\rho_i}} \times \sum_{l=-\infty}^{+\infty} \frac{1}{2\pi} \mathrm{e}^{\mathrm{i}(l-\delta/2\pi)\Delta\theta+\mathrm{i}T(l-\delta/2\pi)(\omega/\alpha)} \times \int D\rho \exp \mathrm{i}\left\{ \int_0^T \mathrm{d}t \left[\frac{m}{2}\rho^2 - \frac{1}{2m\rho^2}\left(\frac{l^2}{\alpha^2} - \frac{1}{4} \right) - \frac{m\omega^2}{2}\rho^2 \right] \right\}. \tag{59}$$

One finds here also the expression of the relative propagator of the inverse quadratic oscillator whose result is

$$K(f,i;T) = \frac{1}{\alpha^2}\frac{m\omega}{\mathrm{i}\sin\omega T} \times \sum_{l=-\infty}^{+\infty} \frac{1}{2\pi} \exp\left(\mathrm{i}\left(l-\delta/2\pi\right)\Delta\theta + \mathrm{i}T\left(l-\delta/2\pi\right)\frac{\omega}{\alpha} \right) \times \exp\left[\frac{\mathrm{i}m\omega}{2\alpha^2}\left(r_f^2 + r_i^2 \right) \cot\omega T \right] \times I_{|l-\delta/2\pi|/\alpha}\left(-\frac{\mathrm{i}m\omega}{\alpha^2\sin\omega T} r_f r_i \right). \tag{60}$$

To extract the wave functions from the previous result and the corresponding spectrum energy, let us use the following formula [10] to separate r_f, r_i and T:

$$\frac{(xyz)^{-\nu/2}}{1-z}\exp\left[-z\frac{x+y}{1-z}\right]I_\nu\left[\frac{2\,(xyz)^{1/2}}{1-z}\right] = \sum_{n=0}^{+\infty}\frac{n!}{\Gamma\left(n+\nu+1\right)}L_n^\nu(x)L_n^\nu(y)z^n, \quad (61)$$

which is valid for $\mid z\mid< 1$.

After choosing the appropriate parameters of the previous formula, the propagator takes the following form:

$$K(f,i;T) = \sum_{n=0}^{+\infty}\sum_{l=-\infty}^{+\infty}\mathrm{e}^{-\mathrm{i}\omega(1+2n+|l-\delta/2\pi|/\alpha-((l-\delta/2\pi))/\alpha)T} \times \Psi_{l,n}\left(f\right)\Psi_{l,n}\left(i\right), \quad (62)$$

which enables us to deduce the wave functions and the corresponding energy spectrum:

$$E_{n,l} = \omega\left(1+2n+\frac{\mid l-\delta/2\pi\mid}{\alpha}-\frac{(l-\delta/2\pi)}{\alpha}\right), \quad (63)$$

and

$$\Psi_{l,n}\left(r,\theta\right) = \sqrt{\frac{m\omega n!}{\pi\alpha^2\Gamma\left(n+1+\dfrac{\mid l-\delta/2\pi\mid}{\alpha}\right)}} \times \mathrm{e}^{\mathrm{i}l\theta-\mathrm{i}(\delta/(2\pi))\theta}\left(\frac{m\omega r^2}{\alpha}\right)^{|l-\delta/2\pi|/(2\alpha)} \times \exp\left(-\frac{m\omega r^2}{2\alpha}\right)L_n^{|l-\delta/2\pi|/(\alpha)}\left(\frac{m\omega r^2}{\alpha}\right). \quad (64)$$

In the same manner as in the free case, it is readily seen that the propagator for this case is also made up of two elementary propagators, one discrete and the other continuous. In order to do this let us compute the compact expression of the propagator following the same procedure as in the free case where we change $\Delta\theta\to\Delta\theta+(\omega T/\alpha)$. The result becomes (see (65) on top of the next page) where $-1/(2\alpha)+\Delta\theta/(2\pi)+\omega T/(2\pi\alpha)\leq n\leq 1/(2\alpha)+\Delta\theta/(2\pi)+\omega T/(2\pi\alpha)$.

2.2 Case of harmonic oscillator

In the case of the harmonic oscillator the potential is given by

$$V(\boldsymbol{x}) = \frac{m\omega^2}{2}(x^2+y^2+z^2) \quad \text{and} \quad \boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{0}. \quad (66)$$

The corresponding propagator is given according to the formula (14) where one replaces the potential $V(\boldsymbol{x})$ by

$$\frac{m\omega^2}{2}\left(x^2+y^2+(\alpha^{-2}-1)(x^2+y^2)\right) = \frac{m\omega^2}{2}\rho^2. \quad (67)$$

The expression of the propagator is, in this case, the same as that used in the free case except that there is now an additional term representing the central harmonic potential. Then the relative expression of the propagator becomes

$$K(f;i;T) = \sum_{l=-\infty}^{+\infty}\frac{1}{2\pi\alpha^2}\mathrm{e}^{\mathrm{i}(l-\delta/2\pi)\Delta\theta}\frac{1}{\sqrt{\rho_f\rho_i}} \times\int D\rho\exp\left\{\mathrm{i}\int_0^T\left[\frac{m}{2}\rho^2-\frac{1}{2m\rho^2}\left(\frac{(l-\delta/2\pi)^2}{\alpha^2}-\frac{1}{4}\right) - \frac{m\omega^2}{2}\rho^2\right]\right\}. \quad (68)$$

The radial part can easily be integrated and deduced directly from the expression of the relative propagator of the inverse quadratic oscillator. In this case, the propagator is written as follows:

$$K(f,i;T) = \frac{m\omega}{\mathrm{i}\alpha^2\sin\omega T} \times\sum_{l=-\infty}^{+\infty}\frac{\mathrm{e}^{\mathrm{i}(l-\delta/2\pi)\Delta\theta}}{2\pi}\exp\left[\frac{\mathrm{i}m\omega}{2\alpha^2}\left(r_f^2+r_i^2\right)\cot\omega T\right] \times I_{|l-\delta/2\pi|/\alpha}\left(-\frac{\mathrm{i}m\omega r_f r_i}{\alpha^2\sin\omega T}\right); \quad (69)$$

by applying the same formula (62), the expression of the propagator will be simplified to

$$K(f;i;T) = \sum_{n=0}^{+\infty}\sum_{l=-\infty}^{+\infty}\mathrm{e}^{-\mathrm{i}\omega(1+2n+|l-\delta/2\pi|/\alpha)T} \times\Psi_{l,n}\left(r_f,\theta_f\right)\Psi_{l,n}\left(r_i,\theta_i\right), \quad (70)$$

which also enables us to extract the energy spectrum:

$$E_{n,l} = \omega\left(1+2n+\frac{\mid l-\delta/2\pi\mid}{\alpha}\right), \quad (71)$$

as well as the wave functions

$$\Psi_{l,n}\left(r,\theta\right) = \sqrt{\frac{m\omega n!}{\pi\alpha^2\Gamma\left(n+1+\dfrac{\mid l-\delta/2\pi\mid}{\alpha}\right)}}\mathrm{e}^{\mathrm{i}(l-\delta/2\pi)\theta} \times\left(\frac{m\omega r^2}{\alpha^2}\right)^{|l-\delta/2\pi|/(2\alpha)}\mathrm{e}^{-(m\omega r^2)/(2\alpha^2)} \times L_n^{|l-\delta/2\pi|/\alpha}\left(\frac{m\omega r^2}{\alpha^2}\right). \quad (72)$$

Following the same method as in the free case we can easily deduce the compact form of the propagator for this case as follows:

$$K(f,i;T) = \frac{m\omega}{2\pi\mathrm{i}\alpha\sin\omega T}\exp\left(\frac{\mathrm{i}m\omega}{2\alpha^2}\left(r_f^2+r_i^2\right)\cot\omega T\right) \times\left\{\sum_n\mathrm{e}^{-\mathrm{i}\delta n}\exp\left[-\frac{\mathrm{i}m\omega r_f r_i}{\alpha^2\sin\omega T}\cos\left(2n\pi\alpha-\alpha\Delta\theta\right)\right]\right.$$

$$K(f,i;T)=\frac{m\omega}{2\pi\mathrm{i}\alpha\sin\omega T}\exp\left(\frac{\mathrm{i}m\omega}{2\alpha^2}\left(r_f^2+r_i^2\right)\cot\omega T\right)\left\{\sum_n \mathrm{e}^{-\mathrm{i}\delta n}\exp\left[-\frac{\mathrm{i}m\omega r_f r_i}{\alpha^2\sin\omega T}\cos\left(2n\pi\alpha-\alpha\Delta\theta-\omega T\right)\right]\right.$$
$$-\frac{\mathrm{e}^{-\mathrm{i}\{\delta/2\pi\}(\Delta\theta+\omega T)}}{2\alpha\pi\mathrm{i}}\int_{0+}^{+\infty}\mathrm{d}t\mathrm{e}^{(\mathrm{i}m\omega r_f r_i)/(\alpha^2\sin\omega T)\cosh t}$$
$$\times\left[\mathrm{e}^{-\mathrm{i}\delta/(2\alpha)+\mathrm{i}[\delta/2\pi](\pi/\alpha)}\left(\frac{\mathrm{e}^{t/\alpha}\sinh\left(\{\delta/2\pi\}\frac{t}{\alpha}\right)+\sinh\left(\mathrm{i}\left(\Delta\theta+\frac{\omega T}{\alpha}+\frac{\pi}{\alpha}\right)-\{\delta/2\pi\}\frac{t}{\alpha}\right)}{\cosh\left(\frac{t}{\alpha}\right)-\cos\left(\Delta\theta+\frac{\omega T}{\alpha}+\frac{\pi}{\alpha}\right)}-\mathrm{e}^{\{\delta/2\pi\}(t/\alpha)}\right)\right.$$
$$\left.\left.-\ \mathrm{e}^{\mathrm{i}\delta/(2\alpha)-\mathrm{i}[\delta/2\pi](\pi/\alpha)}\left(\frac{\mathrm{e}^{t/\alpha}\sinh\left(\{\delta/2\pi\}\frac{t}{\alpha}\right)+\sinh\left(\mathrm{i}\left(\Delta\theta+\frac{\omega T}{\alpha}-\frac{\pi}{\alpha}\right)-\{\delta/2\pi\}\frac{t}{\alpha}\right)}{\cosh\left(\frac{t}{\alpha}\right)-\cos\left(\Delta\theta+\frac{\omega T}{\alpha}-\frac{\pi}{\alpha}\right)}-\mathrm{e}^{\{\delta/2\pi\}(t/\alpha)}\right)\right]\right\},\quad(65)$$

$$-\frac{\mathrm{e}^{-\mathrm{i}\{\delta/2\pi\}\Delta\theta}}{2\pi\mathrm{i}\alpha}\int_{0+}^{+\infty}\mathrm{d}t\exp\left(\frac{\mathrm{i}m\omega r_f r_i}{\alpha^2\sin\omega T}\cosh y\right)$$
$$\times\left[\mathrm{e}^{-\mathrm{i}\delta/(2\alpha)+\mathrm{i}[\delta/2\pi](\pi/\alpha)}\right.$$
$$\times\left(\frac{\mathrm{e}^{t/\alpha}\sinh\left(\{\delta/2\pi\}\frac{t}{\alpha}\right)+\sinh\left(\mathrm{i}\left(\Delta\theta+\frac{\pi}{\alpha}\right)-\{\delta/2\pi\}\frac{t}{\alpha}\right)}{\cosh\left(\frac{t}{\alpha}\right)-\cos\left(\Delta\theta+\frac{\pi}{\alpha}\right)}\right.$$
$$\left.-\ \exp\left(\{\delta/2\pi\}\frac{t}{\alpha}\right)\right)-\mathrm{e}^{\mathrm{i}\delta/(2\alpha)-\mathrm{i}[\delta/2\pi](\pi/\alpha)}$$
$$\times\left(\frac{\mathrm{e}^{t/\alpha}\sinh\left(\{\delta/2\pi\}\frac{t}{\alpha}\right)+\sinh\left(\mathrm{i}\left(\Delta\theta-\frac{\pi}{\alpha}\right)-\{\delta/2\pi\}\frac{t}{\alpha}\right)}{\cosh\left(\frac{t}{\alpha}\right)-\cos\left(\Delta\theta-\frac{\pi}{\alpha}\right)}\right.$$
$$\left.\left.\left.-\ \exp\left(\{\delta/2\pi\}\frac{t}{\alpha}\right)\right)\right]\right\},\quad(73)$$

where $-1/(2\alpha)+\Delta\theta/(2\pi)\leq n\leq\ 1/(2\alpha)+\Delta\theta/(2\pi)$.

2.3 Case of Coulomb potential

In this case

$$V(\boldsymbol{x})=-\frac{\mathrm{e}^2}{\sqrt{(x^2+y^2+z^2)}}\quad\text{and}\quad\boldsymbol{A}(\boldsymbol{x})=\boldsymbol{0},\quad(74)$$

and the propagator is given according to (14), where one replaces the potential $V(\boldsymbol{x})$ by

$$\frac{-\mathrm{e}^2}{\sqrt{x^2+y^2+(\alpha^{-2}-1)(x^2+y^2)}}=\frac{e^2}{\rho}.\quad(75)$$

Let us eliminate the angle φ by doing the relative integration. Thus, the propagator is decomposed into the partial series

$$K(\rho_f,\varphi_f,\rho_i,\varphi_i;T)=\sum_{l=-\infty}^{+\infty}\frac{1}{2\pi\alpha^2}\mathrm{e}^{\mathrm{i}((l/\alpha)-(\delta/(2\pi\alpha)))(\varphi_f-\varphi_i)}K_l(\rho_f,\rho_i;T),\quad(76)$$

where $K_l(\rho_f,\rho_i;T)$ is given by

$$K_l(\rho_f,\rho_i;T)=\frac{1}{\sqrt{\rho_f\rho_i}}\int D\rho\exp\mathrm{i}\left\{\int_0^T\mathrm{d}t\right.$$
$$\left.\times\left[\frac{m}{2}\dot{\rho}^2-\frac{1}{2m\rho^2}\left(\frac{(l-\delta/2\pi)^2}{\alpha^2}-\frac{1}{4}\right)+\frac{e^2}{\rho}\right]\right\},\quad(77)$$

which is the radial propagator related to the hydrogen atom. Using a Duru–Kleinert space-time transformation [8], this propagator is reduced to the radial oscillator which is calculable. For this case, the result of the Green function is as follows (for $\rho_f>\rho_i$):

$$G(f;i;E)=\int_0^\infty\mathrm{d}T\exp(-\mathrm{i}ET)K(f,i;T)$$
$$=\frac{-\mathrm{i}m}{k}\frac{1}{\sqrt{\rho_f\rho_i}}\sum_{l=-\infty}^{+\infty}\frac{\mathrm{e}^{\mathrm{i}((l/\alpha)-(\delta/(2\pi\alpha)))(\varphi_f-\varphi_i)}}{2\pi\alpha^2}$$
$$\times\frac{\Gamma\left(-\nu+\frac{|\,l-\delta/2\pi\,|}{\alpha}+\frac{1}{2}\right)}{\Gamma\left(\frac{2\,|\,l-\delta/2\pi\,|}{\alpha}+1\right)}$$
$$\times M_{\nu,|l-\delta/2\pi|/\alpha}\left(2k\rho_i\right)W_{\nu,|l-\delta/2\pi|/\alpha}\left(2k\rho_f\right),\quad(78)$$

where

$$k=\sqrt{-2mE}\quad\text{and}\quad\nu=e^2\sqrt{\frac{m}{-2E}}.\quad(79)$$

From the poles of the gamma function

$$\Gamma\left(-\nu+(|\,l-\delta/2\pi\,|)/(\alpha)+(1/2)\right)$$

we have

$$-\nu+\frac{|\,l-\delta/2\pi\,|}{\alpha}+\frac{1}{2}=-n;\quad(80)$$

the energy spectrum is deduced to be

$$E_{n,l}=\frac{-e^4 m}{2\left(n+\dfrac{\mid l-\delta/2\pi\mid}{\alpha}+\dfrac{1}{2}\right)^2}. \tag{81}$$

In the vicinity of the poles the function gamma behaves as follows:

$$\begin{aligned}&\Gamma\left(-\nu+\frac{\mid l-\delta/2\pi\mid}{\alpha}+\frac{1}{2}\right) \\ &\approx\frac{-(-1)^n}{n!}\frac{k^2}{m\left(n+\dfrac{\mid l-\delta/2\pi\mid}{\alpha}+\dfrac{1}{2}\right)}\frac{1}{E-E_n}.\end{aligned} \tag{82}$$

The discrete part of the Green function is thus

$$\begin{aligned}G(f;i;E)=&\frac{1}{\sqrt{\rho_f\rho_i}}\sum_{l=-\infty}^{+\infty}\sum_{n=0}^{+\infty}\frac{\mathrm{e}^{\mathrm{i}((l/\alpha)-(\delta/(2\pi\alpha)))(\varphi_f-\varphi_i)}}{2\pi\alpha^2} \\ &\times\frac{\mathrm{i}k}{\left(n+\dfrac{\mid l-\delta/2\pi\mid}{\alpha}+\dfrac{1}{2}\right)}\frac{1}{\Gamma\left(\dfrac{2\mid l-\delta/2\pi\mid}{\alpha}+1\right)} \\ &\times\frac{(-1)^n}{n!}\frac{1}{E-E_n}M_{\nu,|l-\delta/2\pi|/\alpha}\left(2k\rho_i\right)W_{\nu,|l-\delta/2\pi|/\alpha}\left(2k\rho_f\right)\end{aligned} \tag{83}$$

$$\begin{aligned}=&\sum_{l=-\infty}^{+\infty}\sum_{n=0}^{+\infty}\frac{1}{2\pi\alpha^2}\frac{\mathrm{e}^{\mathrm{i}((l/\alpha)-(\delta/(2\pi\alpha)))(\varphi_f-\varphi_i)}}{E-E_n} \\ &\times\frac{2\mathrm{i}k^2 n!\mathrm{e}^{-k\rho_f}\mathrm{e}^{-k\rho_i}}{\left(n+\dfrac{\mid l-\delta/2\pi\mid}{\alpha}+\dfrac{1}{2}\right)\Gamma\left(\dfrac{2\mid l-\delta/2\pi\mid}{\alpha}+1+n\right)} \\ &\times\left(2k\rho_f\right)^{|l-\delta/2\pi|/\alpha}\left(2k\rho_i\right)^{|l-\delta/2\pi|/\alpha} \\ &\times L_n^{2|l-\delta/2\pi|/\alpha}(2k\rho_f)L_n^{2|l-\delta/2\pi|/\alpha}(2k\rho_i).\end{aligned} \tag{84}$$

Returning to the old variables, the final expression of the Green function will then be written

$$\begin{aligned}G(f,i;E)=&\sum_{l=-\infty}^{+\infty}\sum_{n=0}^{+\infty}\frac{1}{2\pi\alpha^2}\mathrm{e}^{\mathrm{i}(l-\delta/2\pi)\Delta\theta}\frac{1}{E-E_n}\frac{2\mathrm{i}k^3 n!}{me^2} \\ &\times\frac{\mathrm{e}^{-kr_f/\alpha}\mathrm{e}^{-kr_i/\alpha}}{\Gamma\left(\dfrac{2\mid l-\delta/2\pi\mid}{\alpha}+1+n\right)} \\ &\times\left(\frac{2kr_f}{\alpha}\right)^{|l-\delta/2\pi|/\alpha}\left(\frac{2kr_i}{\alpha}\right)^{|l-\delta/2\pi|/\alpha} \\ &L_n^{2|l-\delta/2\pi|/\alpha}\left(\frac{2kr_f}{\alpha}\right)L_n^{2|l-\delta/2\pi|/\alpha}\left(\frac{2kr_i}{\alpha}\right).\end{aligned} \tag{85}$$

The bound wave functions are thus

$$\begin{aligned}\Psi_{n,l}(r,\theta)=&\sqrt{\frac{2k^3 n!}{\alpha^2 me^2\Gamma\left(\dfrac{2\mid l-\delta/2\pi\mid}{\alpha}+1+n\right)}} \\ &\times\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}(l-\delta/2\pi)\theta}\mathrm{e}^{-kr/\alpha}\left(\frac{2kr}{\alpha}\right)^{|l-\delta/2\pi|/\alpha} \\ &\times L_n^{2|l-\delta/2\pi|/\alpha}\left(\frac{2kr}{\alpha}\right).\end{aligned} \tag{86}$$

As the kernel (78) is calculable and is expressed with the Bessel functions, the Green function takes the following integral form

$$\begin{aligned}G(\boldsymbol{r}_f,\boldsymbol{r}_i;E)=&\sum_{l=-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}(l-\delta/2\pi)\Delta\theta}\int_0^\infty \mathrm{d}S\frac{m\omega\exp\left(4\mathrm{i}e^2S\right)}{\alpha^2\pi\mathrm{i}\sin\omega S} \\ &\times\exp\left[\frac{\mathrm{i}m\omega}{2\alpha}\left(r_f+r_i\right)\cot\omega S\right] \\ &\times I_{2|l-\delta/2\pi|/\alpha}\left(\frac{m\omega}{\mathrm{i}\alpha\sin\omega S}\sqrt{r_f r_i}\right).\end{aligned} \tag{87}$$

Finally, following step by step the free case, we readily write for this Green function the following compact form:

$$\begin{aligned}G(\boldsymbol{r}_f,\boldsymbol{r}_i,;E)=&\frac{m\omega}{2\pi\alpha\mathrm{i}}\int_0^\infty \mathrm{d}S\frac{\exp\left(4\mathrm{i}e^2S\right)}{\sin\omega S} \\ &\times\exp\left(\frac{\mathrm{i}m\omega}{2\alpha}\left(r_f+r_i\right)\cot\omega S\right) \\ &\times\left\{\sum_n \mathrm{e}^{-\mathrm{i}\delta n}\exp\left[-\frac{\mathrm{i}m\omega\sqrt{r_f r_i}}{\alpha\sin\omega S}\cos\left(n\pi\alpha-\frac{1}{2}\alpha\Delta\theta\right)\right]\right. \\ &-\frac{\mathrm{e}^{-\mathrm{i}\{\delta/2\pi\}\Delta\theta}}{\pi\mathrm{i}\alpha}\int_{0+}^{+\infty}\mathrm{d}t\mathrm{e}^{(\mathrm{i}m\omega\sqrt{r_f r_i})/(\alpha\sin\omega S)\cosh t} \\ &\times\left[\mathrm{e}^{-\mathrm{i}(\delta/\alpha)+\mathrm{i}[\delta/2\pi](2\pi/\alpha)}\right. \\ &\left(\frac{\mathrm{e}^{2t/\alpha}\sinh\left(\{\delta/2\pi\}\dfrac{2t}{\alpha}\right)+\sinh\left(\mathrm{i}\left(\Delta\theta+\dfrac{2\pi}{\alpha}\right)-\{\delta/2\pi\}\dfrac{2t}{\alpha}\right)}{\cosh\left(\dfrac{t}{\alpha}\right)-\cos\left(\Delta\theta+\dfrac{\pi}{\alpha}\right)}\right. \\ &\left.-\ \mathrm{e}^{\{\delta/2\pi\}(2t/\alpha)}\right)-\mathrm{e}^{\mathrm{i}(\delta/(2\alpha))-\mathrm{i}[\delta/2\pi](2\pi/\alpha)} \\ &\left(\frac{\mathrm{e}^{2t/\alpha}\sinh\left(\{\delta/2\pi\}\dfrac{2t}{\alpha}\right)+\sinh\left(\mathrm{i}\left(\Delta\theta-\dfrac{2\pi}{\alpha}\right)-\{\delta/2\pi\}\dfrac{2t}{\alpha}\right)}{\cosh\left(\dfrac{2t}{\alpha}\right)-\cos\left(\Delta\theta-\dfrac{2\pi}{\alpha}\right)}\right. \\ &\left.\left.\left.-\ \mathrm{e}^{\{\delta/2\pi\}(2t/\alpha)}\right)\right]\right\},\end{aligned} \tag{88}$$

where $-1/\alpha+\Delta\theta/(2\pi)\leq n\leq 1/\alpha+\Delta\theta/(2\pi)$.

3 Conclusion

The dynamics of the particle constrained to move on the cone surface and subjected to an external potential ($\boldsymbol{A}(\boldsymbol{x})$,

$V(\boldsymbol{x})$) has been studied via the path integral according to the technique of the constraints of Faddeev–Senjanovic. The propagator, which was expressed in the covering space, was thus evaluated in each following cases: free particle, magnetic field, central harmonic potential and Coulomb potential. The wave functions and the corresponding energy spectrum, in agreement with the references, were correctly deduced. Thus, we showed that the propagator or Green function has a compact form thanks to the integral representation of the Bessel functions.

Finally, the problem of the fall of the particle on the center which is interpreted by the presence of the level energy ($-\infty$), as well as certain physical characteristics such as the scattering amplitude starting from the Green function, are under consideration.

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